The Normal Model & Prior/Posterior Predictive Distributions

STA 702: Lecture 3

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Outline

- Normal Model
- Predictive Distributions
- Prior Predictive; useful for prior elicitation
- Posterior Predictive; predicting/forecasting future events
- Comparing Estimators

Normal Model Setup

- Suppose we have independent observations

$$\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$$

where each $Y_i \mid \theta, \sigma^2 \stackrel{iid}{\sim} \mathsf{N}(\theta, \sigma^2)$

- We will see that it is more convenient to work with $\tau = 1/\sigma^2$ (precision)
- reparameterizing the model for the data we have

$$Y_i \mid \theta, \tau \sim \mathcal{N}(\theta, \tau^{-1})$$

- for simplicity we will treat τ as known initially.
- Need to specify a prior for θ on $\mathbb{R}$

Prior for a Normal Mean

- Natural choice is a Normal/Gaussian distribution (Conjugate prior)

$$\theta \sim \mathsf{N}(\theta_0, 1/\tau_0)$$

- θ_0 is the prior mean - best guess for θ using information other than $\mathbf{y}$
- τ_0 is the prior precision and expresses our certainty about this guess
- one notion of non-informative is to let $\tau_0 \to 0$
- better justification is as Jeffreys' prior (uniform measure) $\pi(\theta) \propto 1$
- parameterization invariant and invariant to location/scale changes in the data (group invariance)

> **Exercise for the Energetic Student**
>
> You should be able to derive Jeffreys prior!

Posterior Distribution (1 observaton)

- Posterior

$$p(\theta \mid y) \propto \exp\left\{-\frac{1}{2}[\tau(y-\theta)^2 + \tau_0(\theta-\theta_0)^2\right\} d\theta$$

- Quadratic in exponential term: $\tau_0(\theta-\theta_0)^2 = \tau_0\theta^2 - 2\tau_0\theta_0\theta + \tau_0\theta_0^2$
  - Expand quadratics, regroup and read off precision from quadtric term in θ and mean from linear term in θ
- posterior precision is the sum of prior precision and data precision $\tau_0 + \tau$
- posterior mean $\hat{\theta} = \frac{\tau_0}{\tau_0+\tau}\theta_0 + \frac{\tau}{\tau_0+\tau}y$; precision weighted average of prior mean and MLE
- conjugate family updating $\theta \mid y \sim \textsf{N}\left(\hat{\theta}, \frac{1}{\tau_0+\tau}\right)$

Marginal Distribution

- Recall that the **marginal distribution** is

$$p(y) = p(y_1, \ldots, y_n) = \int_\Theta p(y_1, \ldots, y_n \mid \theta)\pi(\theta)\, d\theta$$

- this is also called the **prior predictive** distribution and is independent of any unknown parameters
- We may care about making predictions before we even see any data.
- This is often useful as a way to see if the sampling distribution or prior we have chosen is appropriate, after integrating out all unknown parameters.

Prior Predictive for a Single Case

$$\begin{aligned} p(y) &\propto \int_{\mathbb{R}} p(y \mid \theta)\pi(\theta)\, d\theta \\ &\propto \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\tau(y-\theta)^2\right\} \exp\left\{-\frac{1}{2}\tau_0(\theta-\theta_0)^2\right\} d\theta \end{aligned}$$

Integration

1. **Expand** quadratics in exp terms
2. **Group** terms with θ^2 and θ
3. Read off **posterior precision** and **posterior mean**
4. **Complete the square**
5. **Integrate** out θ to obtain marginal!

1. Linear combinations of Normals are Normal!

$$Y \stackrel{D}{=} \theta + \epsilon, \quad \epsilon \sim N(0, 1/\tau) \quad \theta \sim N(\theta_0,$$

2. Find Mean of sum
3. Find Variance of sum
4. Marginal $Y \sim N(\theta_0, 1/\tau_0 + 1/\tau)$

Prior Predictive

- marginal distribution for Y (prior predictive)

$$Y \sim \mathsf{N}\left(\theta_0, \frac{1}{\tau_0} + \frac{1}{\tau}\right) \text{ or } \mathsf{N}(\theta_0, \sigma^2 + \sigma_0^2)$$

- two sources of variability: data variability and prior variability
- useful to think about observable quantities when choosing the prior
- sample directly from the prior predictive and assess whether the samples are consistent with our prior knowledge
- if not, go back and modify the prior & repeat
- sequential substitution sampling (repeat T times)
  1. draw $\theta^{(t)} \sim \pi(\theta)$
  2. draw $y^{(t)} \mid \theta^{(t)} \sim p(y \mid \theta^{(t)})$
- takes into account uncertain about θ and variability in Y!

Posterior Updating

- Sequential updating using the previous result as our prior!
- New prior after seeing 1 observation is

$$\textsf{N}(\theta_1, 1/\tau_1)$$

- prior mean weighted average

$$\theta_1 \equiv \frac{\tau_0 \theta_0 + \tau y_1}{\tau_0 + \tau_1}$$

- prior precision after 1 observation

$$\tau_1 \equiv \tau_0 + \tau$$

- prior variance is now $\sigma_1^2 = 1/\tau_1$

Posterior Predictive for y_2 given y_1

- Conditional $p(y_2 \mid y_1) = p(y_2, y_1)/p(y_1)$ (Hard way!)
- Use latent variable representation

$$p(y_2 \mid y_1) = \int_\Theta \frac{p(y_2, \mid \theta) p(y_1 \mid \theta) \pi(\theta) \, d\theta}{p(y_1)}$$

- simplify to previous problem and use results

$$p(y_2 \mid y_1) = \int_\Theta p(y_2 \mid \theta) \pi(\theta \mid y_1) \, d\theta$$

- (Posterior) Predictive

$$Y_2 \mid y_1 \sim \textsf{N}(\theta_1, \sigma^2 + \sigma_1^2)$$

Iterated Expectations

Based on expressions we have an exponential of a quadratic in y_2 so know that distribution is Gaussian

- Find the mean and variance using iterated expectations:
- mean

$$\mathsf{E}[Y_2 \mid y_1] = \mathsf{E}_{\theta \mid y_1}[\mathsf{E}_{Y_2 \mid y_1, \theta}[Y_2 \mid y_1, \theta] \mid y_1]$$

- Conditional Variance $\mathsf{Var}[Y_2 \mid y_1]$
- Iterated expectations (prove!)

$$\mathsf{Var}[Y_2 \mid y_1] = \mathsf{E}_{\theta \mid y_1}[\mathsf{Var}_{Y_2 \mid y_1, \theta}[Y_2 \mid y_1, \theta] \mid y_1] + \mathsf{Var}_{\theta \mid y_1}[\mathsf{E}_{Y_2 \mid y_1, \theta}[Y_2 \mid y_1, \theta] \mid y_1]$$

Updated Posterior for θ

$$p(\theta \mid y_1, y_2) \propto p(y_2 \mid \theta) p(y_1 \mid \theta) \pi(\theta)$$

$$p(\theta \mid y_1, y_2) \propto p(y_2 \mid \theta) p(\theta \mid y_1)$$

Apply previous updating rules

- new posterior mean

$$\theta_2 = \frac{\tau_1 \theta_1 + \tau y_2}{\tau_1 + \tau} = \frac{\tau_0 \theta_0 + 2\tau \bar{y}}{\tau_0 + 2\tau}$$

- new precision

$$\tau_2 = \tau_1 + \tau = \tau_0 + 2\tau$$

After n observations

- Posterior for θ

$$\theta \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\frac{\tau_0 \theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau_0 + n\tau}\right)$$

- Posterior Predictive Distribution for Y_{n+1}

$$Y_{n+1} \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\frac{\tau_0 \theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau} + \frac{1}{\tau_0 + n\tau}\right)$$

- Shrinkage of the MLE to the prior mean
- More accurate estimation of θ as $n \to \infty$ (reducible error)
- Cannot reduce the error for prediction Y_{n+1} due to σ^2
- predictive distribution for a next observation given *everything* we know - prior and likelihood

Results with Jeffreys' Prior

- What if $\tau_0 \to 0$? (or $\sigma_0^2 \to \infty$)
- Prior predictive $\textsf{N}(\theta_0, \sigma_0^2 + \sigma^2)$ (not proper in the limit)
- Posterior for θ (formal posterior)

$$\theta \mid y_1, \ldots, y_n \sim \textsf{N}\left(\frac{\tau_0 \theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau_0 + n\tau}\right)$$

$$\to \qquad \theta \mid y_1, \ldots, y_n \sim \textsf{N}\left(\bar{y}, \frac{1}{n\tau}\right)$$

- Recovers the MLE as the posterior mode!
- Posterior variance of $\theta = \sigma^2/n$ (same as variance of the MLE)

Posterior Predictive Distribution

- Posterior predictive distribution for Y_{n+1}

$$Y_{n+1} \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\frac{\tau_0\theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau} + \frac{1}{\tau_0 + n\tau}\right)$$

- Under Jeffreys' prior

$$Y_{n+1} \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\bar{y}, \sigma^2(1 + \frac{1}{n})\right)$$

- Captures extra uncertainty due to not knowing θ (compared to plug-in approach where we plug in MLE in sampling model!

Comparing Estimators

- Expected loss (from frequentist perspective) of using Bayes Estimator
- Posterior mean is optimal under squared error loss (min Bayes Risk) [also absolute error loss]
- Compute Mean Square Error (or Expected Average Loss)

$$\mathsf{E}_{\bar{y}|\theta}\left[\left(\hat{\theta}-\theta\right)^2 \mid \theta\right]$$

$$= \mathsf{Bias}(\hat{\theta})^2 + \mathsf{Var}(\hat{\theta})$$

- For the MLE $\bar{Y}$ this is just the variance of $\bar{Y}$ or σ^2/n

MSE for Bayes

- Frequentist Risk

$$\mathsf{E}_{\bar{y}|\theta}\left[\left(\hat{\theta}-\theta\right)^2 \mid \theta\right] = \mathsf{MSE} = \mathsf{Bias}(\hat{\theta})^2 + \mathsf{Var}(\hat{\theta})$$

- Bias of Bayes Estimate

$$\mathsf{E}_{\bar{Y}|\theta}\left[\frac{\tau_0\theta_0 + \tau n\bar{Y}}{\tau_0 + \tau n}\right] = \frac{\tau_0(\theta_0 - \theta)}{\tau_0 + \tau n}$$

- Variance

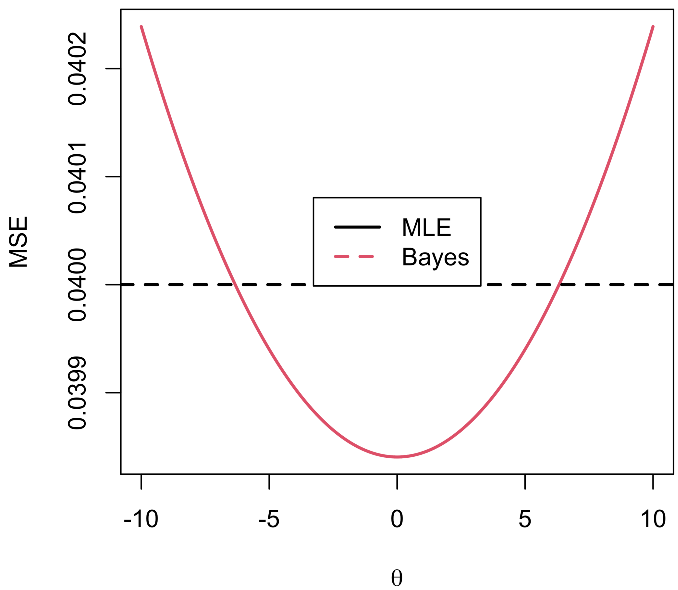
$$\mathsf{Var}\left(\frac{\tau_0\theta_0 + \tau n\bar{Y}}{\tau_0 + \tau n} - \theta \mid \theta\right) = \frac{\tau n}{(\tau_0 + \tau n)^2}$$

- (Frequentist) expected Loss when truth is θ

$$\mathsf{MSE} = \frac{\tau_0^2(\theta - \theta_0)^2 + \tau n}{(\tau_0 + \tau n)^2}$$

Plot

Behavior ?



Updating with n Observations

- Can update sequentially as before -or-
- We can use the $\mathcal{L}(\theta)$ based on n observations and repeat completing the square with the original prior $\theta \sim \mathsf{N}(\theta_0, 1/\tau_0)$
- same answer!
- The likelihood for θ is proportional to the sampling model

$$p(y \mid \theta, \tau) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\tau(y_i - \theta)^2\right\}$$

> **Exercise**
>
> Rewrite in terms of sufficient statistics!

Exercises for Practice

Exercise 1

Use $\mathcal{L}(\theta)$ based on n observations and $\pi(\theta)$ to find $\pi(\theta \mid y_1, \ldots, y_n)$ based on the sufficient statistics

Exercise 2

Use $\pi(\theta \mid y_1, \ldots, y_n)$ to find the posterior predictive distribution for Y_{n+1}

Simplification

$$
\begin{aligned}
\mathcal{L}(\theta) &\propto \tau^{\frac{n}{2}} \, \exp\left\{-\frac{1}{2}\tau\sum_{i=1}^{n}(y_i-\theta)^2\right\} \\
&\propto \tau^{\frac{n}{2}} \, \exp\left\{-\frac{1}{2}\tau\sum_{i=1}^{n}\left[(y_i-\bar{y})-(\theta-\bar{y})\right]^2\right\} \\
&\propto \tau^{\frac{n}{2}} \, \exp\left\{-\frac{1}{2}\tau\left[\sum_{i=1}^{n}(y_i-\bar{y})^2+\sum_{i=1}^{n}(\theta-\bar{y})^2\right]\right\} \\
&\propto \tau^{\frac{n}{2}} \, \exp\left\{-\frac{1}{2}\tau\left[\sum_{i=1}^{n}(y_i-\bar{y})^2+n(\theta-\bar{y})^2\right]\right\} \\
&\propto \tau^{\frac{n}{2}} \, \exp\left\{-\frac{1}{2}\tau s^2(n-1)\right\} \, \exp\left\{-\frac{1}{2}\tau n(\theta-\bar{y})^2\right\}
\end{aligned}
$$